Question 1

(1) Suppose u, v, x, y, z are integers such that $u \equiv v \pmod{x}$ and $y \equiv z \pmod{x}$.
Prove that $uz \equiv vy \pmod{x}$? Please show all workings. [5 marks]

(2) Convert $E7_{16}$ into its binary equivalent. [1 marks]

(3) Convert 10011_2 into its octal equivalent. [1 marks]

(4) Convert 127_8 into its decimal equivalent. [1 marks]

(5) Use binary arithmetic to perform the calculation $1011_2 \times 1010_2$.
Show all workings. [2 marks]

Question 2

(1) Translate the following inference into propositional logic. [3 marks]

> If today is Wednesday, then I have a test in Math or a test in Eng.
> If my Eng professor is sick, then I will not have a test in Eng.
> Today is Wednesday and my Eng professor is sick. Therefore I have a test in Math.

Is the inference correct? Justify your answer. [1 marks]

(2) Using the predicates Bx ("x is a barber in Podunk") and Σxy ("x shaves y"), translate the following argument into a sentence of the predicate calculus. [3 marks]

> Any barber in Podunk shaves exactly those individuals who do not shave themselves.
> Therefore, there is no barber in Podunk.

Use an unsigned tableau to test this argument for logical validity. [3 marks]

Question 3

(1) Prove that if A, B, C are sets, then $C - (A \cup B) = (C - A) \cap (C - B)$.
Please show all workings. [4 marks]

(2) Prove that at a cocktail party with six or more people, there are either three mutual acquaintances or three mutual strangers. [6 marks]

Question 4

Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7, 9\}$. Suppose that P is a relation from A to B defined as follows. In each case, give the elements of P. [10 marks]

(1) $a \text{ P } b \Leftrightarrow a \equiv b \pmod{4}$

(2) $a \text{ P } b \Leftrightarrow 2 | (a + b)$

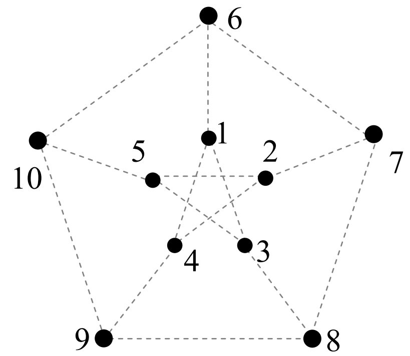
(3) $a \text{ P } b \Leftrightarrow a + b + 1 \equiv 0 \pmod{3}$

(4) $a \text{ P } b \Leftrightarrow 2a^2 \geq b$

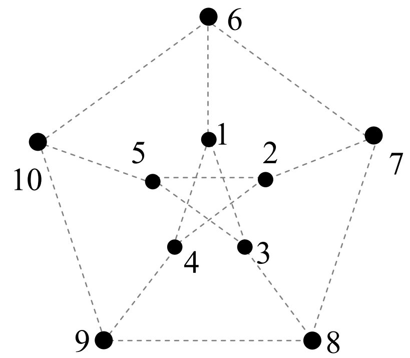
(5) $a \text{ P } b \Leftrightarrow a = b \text{ or } a - 1 = b$

Question 5

(1) Find a breadth-first search tree in the graph on the right, the Peterson graph, starting at vertex 1 as root. Draw the tree edges in solid lines, and draw arrows on them to indicate the parent of each non-root vertex. List the order in which the vertices enter the queue. [4 marks]



(2) What is the least number of edges we can delete from the above graph, so as to make it bi-partite? Justify your answer. In the copy of the graph on the right, make a cross on the edges you delete and draw the edges that you keep in solid. [6 marks]



Question 6

Consider the following recursive algorithm, named *exp*, which takes a real number x and a natural number y as inputs and is supposed to output the value x^y.

Algorithm $exp(x, y)$
  if $y = 0$ then return (1)
  else
    $z := exp(x, \left\lfloor \frac{y}{2} \right\rfloor)$
    if y is even then return $(z \times z)$
    else return $(x \times z \times z)$

(1) Hand-turn the algorithm to compute *exp*(3, 7). Please show all your workings. [4 marks]

(2) Prove by induction on y that the algorithm is correct. [6 marks]

Question 7

Let α be the permutation of $\{1, 2, \ldots, 9\}$ given by $1\alpha = 2$, $2\alpha = 6$, $3\alpha = 3$, $4\alpha = 5$, $5\alpha = 4$, $6\alpha = 8$, $7\alpha = 1$, $8\alpha = 9$ and $9\alpha = 7$.

(1) Write α in two line notation. [3 marks]

(2) Find a cycle decomposition of α. [3 marks]

(3) Find a cycle decomposition of $\alpha\beta$ with α as above and $\beta = (2, 4, 7, 3)$. [4 marks]

Question 8

(1) How many positive integers that are divisible by 5 and having four distinct digits are there? Please show all your workings. [3 marks]

(2) How many four-digit positive integers have their digits in nondecreasing (e.g., 3359 and 2478) or nonincreasing (e.g., 8660 and 5421) order? You may leave your answer as an unevaluated expression. Please show all your workings. [4 marks]

(3) How many ways can three couples be seated in a row of six seats if each couple is seated together? Please show all your workings. [3 marks]

Question 9

Shuffle a standard 52-card deck, and deal out a hand of 13 cards. You get 4 points for each Ace in the hand, 3 points for each King, 2 for each Queen, 1 for each Jack, and nothing for the other cards. Let the random variable X denote the total number of points in your hand.

(1) Calculate $\Pr[X = 0]$. You may leave your answer as an unevaluated expression. Show your workings. [3 marks]

(2) Calculate $E[X]$. This time we want a number; again, show your workings. [4 marks]

(3) Suppose we tell you that your hand has 6 spades, 2 hearts, 3 diamonds, and 2 clubs. Now consider the expected value of X, conditioned on this fact. Does this number increase, decrease, or stay the same, compared to your answer in part (2)? Briefly justify your answer. [3 marks]

Question 10

(1) You would like to encode a sequence of symbols that come from an alphabet with $d + 3$ symbols. You want to encode symbols a_1, a_2, and a_3 using codewords that are three bits long. You want to encode symbols $a_4, a_5, \ldots, a_{d+3}$ using codewords that are eight bits long. What is the maximum value of d for which this will be possible, if the code must be uniquely decodable? Please show your workings. [5 marks]

(2) Consider a source with source alphabet $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ in which the symbols probabilities are as follows: $p_1 = 0.27$; $p_2 = 0.09$; $p_3 = 0.23$; $p_4 = 0.11$; $p_5 = 0.15$; and $p_6 = 0.15$.

One possible Huffman code is
a_1 : 11
a_2 : 000
a_3 : 01
a_4 : 001
a_5 : 100
a_6 : 101

Compute the expected codeword length for this Huffman code. Please show your workings. [5 marks]