

1. (10%) Consider the linear system $Ax = b$ given by
$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 4 & 1 & p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ k \end{bmatrix}.$$

If the matrix A is not invertible, what is the value of p ? Apply this value of p and find the value of k that makes the linear system $Ax = b$ have a solution.

2. (15%) For the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, find a matrix P and the diagonal matrix

D such that $D = P^{-1}AP$.

3. (15%) Solve the following differential equation $(x^2 + 1)y'' + 4xy' + 2y = 0$.

4. (15%) Solve the following initial value problem $y''(t) + 2ty'(t) - 4y(t) = 1$; $y(0) = 0$, $y'(0) = 0$, using the Laplace transformation.

5. (15%) Solve the integral $\int_{-\infty}^{\infty} \left[\frac{\sin(\omega a)}{\omega a} \right]^2 d\omega$ using the Fourier transformation.

6. (15%) A particle moves once counterclockwise about the triangle with vertices $(0,0)$, $(4,0)$ and $(1,6)$, under the influence of the force $\vec{F} = (xy)\vec{i} + (x)\vec{j}$.

Calculate the integral $\oint_C \vec{F} \cdot d\vec{r} = ?$

7. (15%) Let C is a closed rectangular contour traversed counterclockwise, with vertices at $-1-i$, $3-i$, $3+3i$, $-1+3i$. Find $\oint_C \left(\frac{z+1}{z-1} \right)^3 dz = ?$