1. Single Choice Problems (each sub-problem: 5 points)

(1) Assume that n is an exact power of 2, which is the solution of the recurrence relation shown below?

$$\Gamma(n)=\begin{cases} 2 & \text{if } n=2 \\ 2\Gamma(n/2)+n & \text{if } n=2^k, \forall k>1 \end{cases}$$

(A) $\Gamma(n)=2\lg n$ (B) $\Gamma(n)=n\lg n$ (C) $\Gamma(n)=n\lg\lg n$ (D) $\Gamma(n)=n\lg\lg\lg n$

(2) For non-negative real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, assume $n \geq 1{,}000$, which is the correct inequality?

(A) $\sum_{i=1}^{n}\left(a_i \log\frac{a_i}{b_i}\right) \leq \left(\sum_{i=1}^{n} a_i\right)\log\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$

(B) $\sum_{i=1}^{n}\left(a_i \log\frac{a_i}{b_i}\right) \geq \left(\sum_{i=1}^{n} a_i\right)\log\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$

(C) $\sum_{i=1}^{n}\left(a_i \log\frac{a_i}{b_i}\right) \neq \left(\sum_{i=1}^{n} a_i\right)\log\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$

(3) How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

(A) $\binom{8}{4}$ (B) $2\cdot\binom{8}{4}$ (C) $3\cdot\binom{8}{4}$ (D) $6\cdot\binom{8}{4}$

(4) If a and b are relatively prime, which is the correct formula?

(A) $\gcd(a, b) = \gcd(a \bmod b, b)$

(B) $a^{\phi(b)} \bmod b = 1$, where $\phi(x)$ is the number of positive integers less than x and relatively prime to x

(C) $(a-1)! = 1 \bmod b$

(D) $a^{b-1} = 1 \bmod b$

2. The harmonic numbers H_j, $j = 1, 2, 3, \ldots$, are defined as follows:

$H_j = 1 + 1/2 + 1/3 + \ldots + 1/j$.

Show that $H_{2^n} \geq 1+\frac{n}{2}$,

where n is a nonnegative integer. (15 points)

Accordingly, is H_j a divergent infinite series? (5 points)

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3. The k of r out of n circular reliability system $k \le r \le n$ consists of n components, each of which is either functioning or failed, that are arranged in a circular fashion. The system itself is said to be functional if there is no block of r consecutive components of which at least k are failed.

(1) Is it possible to arrange 39 components, 7 of which are failed, to make a functional 3 of 12 out of 39 circular system? (10 points)

(2) For a fixed integer k, $k \le n$, please specify a sufficient condition on k and r that makes it possible to arrange n components, $n/5$ of which are failed, to make a functional k of r out of n circular system. (10 points)

4. When using a binary search algorithm to find an element e in an n-element list L, how many elements in the list will be examined if the algorithm returns a failure in search of e (i.e., $e \notin L$)? (10 points)

5. If we flip a coin, there is probability p that it comes up heads and probability q that it comes up tails, where $p + q = 1.0$; i.e., this process have just two outcomes. If we toss the coin n times and assume that different coin tosses are always independent. Then what is the chance (in terms of n, k, p, and q) of obtaining exactly k heads in n tosses? (20 points)

6. Which summation formula shown below is wrong? Show the correct expression if there is a need. (10 points)

(A) $\sum_{k=0}^{n} ar^k (r \neq 0) = \frac{ar^{n+1} - a}{r-1}, r \neq 1$

(B) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(C) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

(D) None