1. (25%) Let ***S*** be a stack that stores at most n integers. The operations of the stack are **PUSH**(***S***,x), **POP**(**S**) and **MULTIPOP**(***S***, k). Each of the **PUSH** and **POP** operations takes O(1) time and the definitions for the three operations are given as follows:

> **PUSH**(***S***,x) pushes object x onto stack ***S***.
>
> **POP**(***S***) pops the top of stack ***S*** and returns the popped object.
>
> **MULTIPOP**(***S***, k)
> while not **STACK-EMPTY**(***S***) and $k\neq 0$
> do { **POP**(***S***)
> $k=k-1$ }

Answer the following questions.

(a) (5%) Assume the stack ***S*** is empty initially. Draw a picture of ***S*** after the sequence of operations have been performed: **PUSH**(***S***,a) **PUSH**(***S***,b) **POP**(**S**) **PUSH**(***S***,c).

(b) (5%) Assume the stack ***S*** is empty initially. Draw a picture of ***S*** after the sequence of operations have been performed: **PUSH**(***S***,a) **PUSH**(***S***,b) **PUSH**(***S***,c) **MULTIPOP**(***S***, 2) **PUSH**(***S***,d).

(c) (5%) Assume the stack ***S*** is empty initially. Draw a picture of ***S*** after the sequence of operations have been performed: **PUSH**(***S***,a) **PUSH**(***S***,b) **PUSH**(***S***,c) **MULTIPOP**(***S***, 4) **PUSH**(***S***,d).

(d) (5%) What is the time complexity of **one** stack operation. (Your answer should be as tight as possible)

(e) (5%) Assume the stack ***S*** is empty initially. What is the time complexity of **a sequence** of n stack operations. (Your answer should be as tight as possible)

2. (10%) Solve the following recurrence relations:

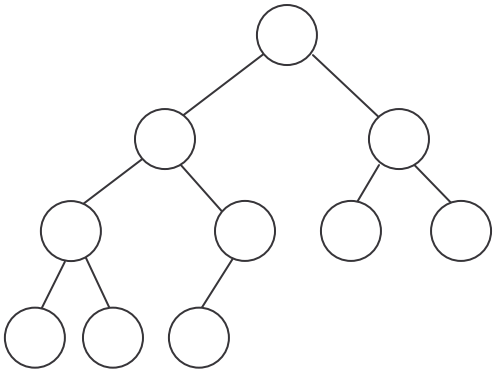
(a) (5%) $f(n)=f(n-1)+2$ for $n>1$, $f(1)=0$

(b) (5%) $f(n)=f(n/2)+n$ for $n>1$, $f(1)=1$ (solve for $n=2^k$)

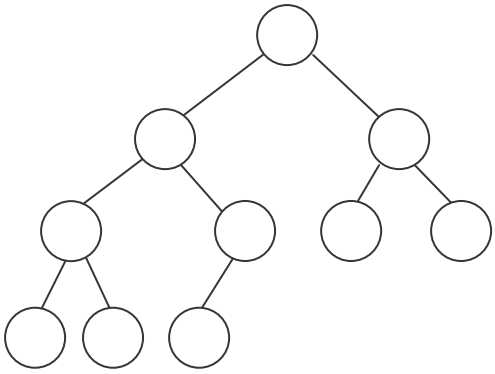
3. (10%) Array **A**[1:n] contains n integers sorted in increasing order and array **B**[1:2n] contains 2n integers sorted in increasing order. Write an algorithm that merges arrays **A** and **B** into array **C** in increasing order with time complexity of $\Theta(n)$.

4. (10%) Show how to sort n integers in the range 0 to n^2-1 in O(n) time ?

5. (20%) Integers 1, 2, 3, 4, 7, 8, 9, 10, 14, and 16 are the keys of the nodes in the following trees. Please assign the keys to the nodes of the trees to satisfy (a) (10%) max-heap property (b) (10%) binary search tree property.



max-heap



binary search tree

6. (15%) There are eleven cities A, B, C, D, E, F, G, H, I, J and K connected by unpaved roads as shown in the following Figure (a1). The lengths of the roads between pairs of towns are represented by edge weights. (a) (10%) Which roads should be paved so that there is a path of paved roads between each pair of towns and the total length of the paved roads should be minimized? (b) (5%) What is the minimal total length of the paved roads?

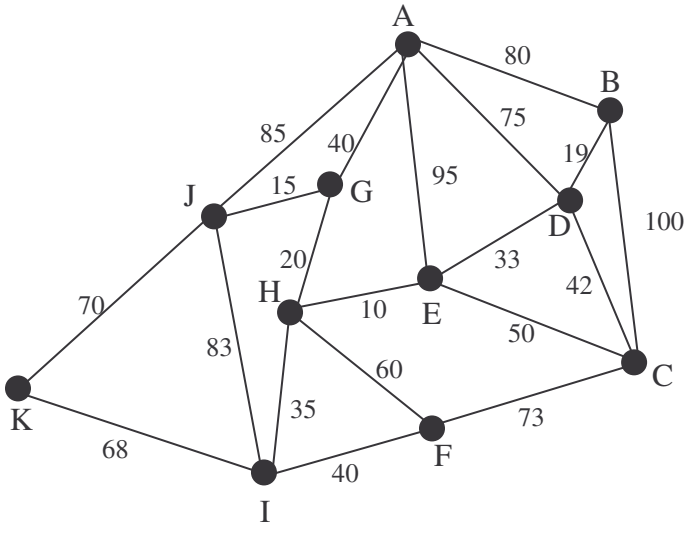


Figure (a1)

7. (10%) Optimal substructure is one of the key ingredients that an optimization problem must have in order for dynamic programming to be applicable. Explain what does it mean that a problem exhibits optimal substructure.